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LETTER TO THE EDITOR

Derrick's theorem in curved space

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Abstract. A class of scalar field variations is found which strengthen the results of Radmore and Stephenson concerning the non-existence of soliton-like solutions of non-linear wave equations in a Reissner–Nordström background.

In a recent letter, Radmore and Stephenson (1978) consider the existence of solutions to the non-linear Klein–Gordon equation

$$\Box\Phi = -\tfrac{1}{2}f'[\Phi] \tag{1}$$

in a Reissner–Nordström background space. They attempt to find a generalisation of the flat space theorem of Derrick (1964), who, by considering a specific variation of Φ, showed that the condition for stability of Φ cannot be simultaneously satisfied with its equation of motion.

In this letter it is shown that in a Reissner–Nordström background the variation of Φ given by Radmore and Stephenson is technically unsatisfactory. A suitable variation of Φ can be found, and for a class of functionals $f[\Phi]$, it is demonstrated that solutions to (1) are physically unrealisable. The results strengthen the conclusions of Radmore and Stephenson.

Assume Φ is invariant under the isometries of the background space, then (1) becomes

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left((r-r_+)(r-r_-)\frac{\mathrm{d}\Phi}{\mathrm{d}r}\right) = \tfrac{1}{2}f'[\Phi] \tag{2}$$

where r_- and r_+ are the locations of the inner and outer event horizons respectively.

Equation (2) implies the variational principle

$$\delta E = 0 \tag{3}$$

for the energy E of the Φ-field exterior to r_+. If we write

$$I_1 = \int_{r_+}^{\infty} (r-r_+)(r-r_-)(\mathrm{d}\Phi/\mathrm{d}r)^2\,\mathrm{d}r \tag{4}$$

$$I_2 = \int_{r_+}^{\infty} f[\Phi]r^2\,\mathrm{d}r \tag{5}$$

so that

$$E = 4\pi(I_1+I_2) \tag{6}$$

we require that both I_1 and I_2 should exist.

Furthermore, stability of Φ requires

$$\delta^2 E > 0. \tag{7}$$

We attempt to find a variation which cannot satisfy both (3) and (7).

As with standard Lagrangian theory, equivalence between the equation of motion and a variational principle inside some volume V assumes the variation vanishes on the boundary of V. In the case where the boundary lies at infinity, this requirement is mitigated by ensuring sufficient asymptotic boundary conditions on Φ. Such is the case for Derrick's theorem.

In Radmore and Stephenson's paper, however, the 3-volume in which the variation takes place is defined by $\infty > r \geqslant r_+$. Consequently, their variation

$$\Phi_\alpha(r) = \Phi(\alpha r) \tag{8}$$

where α is an arbitrary constant, does not vanish on the boundary $r = r_+$, and the equation of motion (2) does not give rise to (3).

For this reason, consider the variation

$$\Phi_\alpha(r) = \Phi(\alpha r - (\alpha - 1) r_+) \tag{9}$$

which satisfies

$$\Phi_\alpha(r_+) = \Phi(r_+) \tag{10}$$

$$\Phi_\alpha(r) = \Phi(\alpha r), \qquad r \gg r_+ \tag{11}$$

$$\Phi_{\alpha=1}(r) = \Phi(r). \tag{12}$$

Equation (10) ensures the variation vanishes on r_+, while (11) ensures equivalence with (8) at large distances from the event horizon.

Putting

$$E_\alpha = 4\pi \int_{r_+}^{\infty} [(r - r_+)(r - r_-)(\mathrm{d}\Phi_\alpha/\mathrm{d}r)^2 + f[\Phi_\alpha] r^2]\, \mathrm{d}r \tag{13}$$

then, using (9), a straightforward calculation gives

$$\mathrm{d}E_\alpha/\mathrm{d}\alpha|_{\alpha=1} = 4\pi \int_{r_+}^{\infty} [(r - r_+)^2 (\mathrm{d}\Phi/\mathrm{d}r)^2 + f[\Phi] r (3r - 2r_+)]\, \mathrm{d}r \tag{14}$$

$$\mathrm{d}^2 E_\alpha/\mathrm{d}\alpha^2|_{\alpha=1} = 8\pi \int_{r_+}^{\infty} f[\Phi](r - r_+)(3r - r_+)\, \mathrm{d}r. \tag{15}$$

Hence, since I_1 exists, (3) and (14) give

$$\int_{r_+}^{\infty} f[\Phi] r (3r - 2r_+)\, \mathrm{d}r < 0 \tag{16}$$

while (7) and (15) give

$$\int_{r_+}^{\infty} f[\Phi](r - r_+)(3r - r_+)\, \mathrm{d}r > 0. \tag{17}$$

Notice that as $r_+ \to 0$ we recover Derrick's result that (16) and (17) are mutually incompatible.

If $f[\Phi] > 0$, we are able to confirm Radmore and Stephenson's conclusion that only trivial solutions to (1) exist. However, we are able to go further than this. If $f[\Phi] < 0$, then, while (3) is satisfied, the stability condition (7) is not, and again only trivial solutions to (1) exist.

If no restriction on the sign of $f[\Phi]$ is made, we may proceed as follows. Since I_2 exists, integration of (16) and (17) by parts gives

$$\int_{r_+}^{\infty} (\mathrm{d}f[\Phi]/\mathrm{d}r)r^2(r-r_+)\,\mathrm{d}r>0 \tag{18}$$

$$\int_{r_+}^{\infty} (\mathrm{d}f[\Phi]/\mathrm{d}r)r(r-r_+)^2\,\mathrm{d}r<0. \tag{19}$$

But since

$$\mathrm{d}f[\Phi]/\mathrm{d}r=f'[\Phi]\,\mathrm{d}\Phi/\mathrm{d}r \tag{20}$$

then, using (2) (which implies (18)), (19) becomes

$$2\int_{r_+}^{\infty}\left(\frac{\mathrm{d}\Phi}{\mathrm{d}r}\right)^2\frac{1}{r}(r-r_+)^2(2r-r_--r_+)\,\mathrm{d}r+\int_{r_+}^{\infty}\frac{\mathrm{d}}{\mathrm{d}r}\left[\left(\frac{\mathrm{d}\Phi}{\mathrm{d}r}\right)^2\right]\frac{(r-r_+)(r-r_-)}{r}\,\mathrm{d}r<0. \tag{21}$$

Since I_1 exists, the second integral in (21) may be integrated by parts. Collecting terms, (21) becomes

$$\int_{r_+}^{\infty}\left(\frac{\mathrm{d}\Phi}{\mathrm{d}r}\right)^2\frac{(r-r_+)^2}{r^2}(r^2-2rr_++r_+r_-)\,\mathrm{d}r<0 \tag{22}$$

so that stability implies

$$r^2-2rr_++r_+r_-<0. \tag{23}$$

For a maximal black hole ($r_+=r_-$), (23) is violated and the field is unstable. For a regular black hole, there is a small region ($2r_+>r>r_+$ for a Schwarzschild space) which gives a negative contribution to (22).

Hence with no restriction on the sign of $f[\Phi]$ no immediate conclusion can be reached as to the stability of Φ.

The author has considered the more general class of variation

$$\Phi_\alpha(r)=\Phi(r+(\alpha-1)g(r-r_+)) \tag{24}$$

for arbitrary functions g satisfying $g(0)=0$, but cannot strengthen the above results.

References

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